CONVEXITY OF INVERSION
FOR POSITIVE OPERATORS ON A HILBERT SPACE

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ABSTRACT

CONVEXITY OF INVERSION FOR POSITIVE OPERATORS ON A HILBERT SPACE. This paper discusses and proves three theorems for positive invertible operators on a Hilbert space. The first theorem gives a comparison of the generalized arithmetic mean, generalized geometric mean, and generalized harmonic mean for positive invertible operators on a Hilbert space. For the second and third theorems each gives three inequalities for positive invertible operators on a Hilbert space that are mutually equivalent.

PRELIMINARIES

Let $X$ be a real or complex vector space. A norm on $X$ is a non-negative real valued function $\| \cdot \|$ on $X$ such that for all scalar $\alpha$ and all $x, y \in X$ the following hold:

(i) $\|x\| \geq 0$; $\|x\| = 0 \iff x = 0$;
(ii) $\|x + y\| \leq \|x\| + \|y\|$;
(iii) $\|\alpha x\| = |\alpha| \|x\|$.

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A real (complex) normed linear space is a real (complex) vector space $X$ together with a specified norm on $X$. On such a space we have a metric $\rho$ defined by

$$\rho(x, y) = \|x - y\|$$

If $X$ is complete in this metric we call $X$ a Banach space. Completeness means that if \{x$_n$\} is a sequence of elements of $X$ such that

$$\lim_{n,m \to \infty} \|x_m - x_n\| = 0$$

there exists an element $x$ in $X$ such that

$$\lim_{n \to \infty} \|x - x_n\| = 0.$$

Let $H$ be a real or complex vector space. An inner product on $H$ is a function $(\ , \ )$ which assigns to each ordered pair of vectors in $H$ a scalar, in such a way that for all $x, y, z \in H$ and all scalar $\alpha$ the following hold:

\begin{itemize}
  \item[(i)] $(x + y, z) = (x, z) + (y, z)$;
  \item[(ii)] $(\alpha x, y) = \alpha(x, y)$;
  \item[(iii)] $(x, y) = (y, x)$;
  \item[(iv)] $(x, x) \geq 0; (x, x) = 0 \iff x = 0.$
\end{itemize}

Such a space $H$, together with a specified inner product on $H$, is called an inner product space. In any inner product space $H$ we have the Cauchy-Schwarz inequality:

$$\|x, y\|^2 \leq (x, x)(y, y), \text{ for all } x, y \in H.$$

From the Cauchy-Schwarz inequality it follows that $\|x\| = (x, x)^{1/2}$ constitutes a norm on $H$. If $H$ is complete in this norm, we call that $H$ is a Hilbert space.

A function $f$ defined on a set $S$ is called to be convex if for each $x, y \in S$ and each $\alpha, 0 \leq \alpha \leq 1$ we have

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y).$$

An operator $A$ on a Hilbert space $H$ is called to be positive if $A(x) > 0$ for every $x \in H$. If $T$ is a positive operator on $H$ then for real numbers $\alpha, \beta$ we define the operator $\alpha T + \beta$ on $H$ by $(\alpha T + \beta)(x) = \alpha T(x) + \beta, x \in H$.

**MAIN RESULT**

The main result of this paper are the following three theorems. The first one gives a comparison of the generalized arithmetic mean, generalized geometric mean, and generalized harmonic mean for positive invertible operators on a Hilbert space.
For the second and third theorems each gives three inequalities for positive invertible operators on a Hilbert space that are mutually equivalent.

**Theorem 1**

Let \( A \) and \( B \) be positive invertible operators on a Hilbert space \( H \). Then for \( 0 \leq \alpha \leq 1 \) the following inequalities hold:

\[
(1-\alpha) A + \alpha B \geq A^{1/2} (A^{-1/2} B A^{-1/2})^{\alpha} A^{1/2} \geq [(1-\alpha) A^{-1} + \alpha B^{-1}]^{1/2}. \tag{1}
\]

**Proof**

Consider the function \( F(x) = \alpha x + 1 - \alpha - x^\alpha \) for positive real numbers \( x \) and \( 0 \leq \alpha \leq 1 \). First we need to show that \( F(x) \) is a nonnegative function on the interval \([0, \infty)\). Recall that \( F(x) \) is continuous on \([0, \infty)\). By routine calculation we get

\[
F(0) = 1 - \alpha \geq 0, \quad F(1) = 0, \quad F'(x) = \alpha (1 - x^{\alpha-1})
\]

and

\[
F'(x) = \alpha (1 - x^{\alpha-1}) \leq 0, \quad 0 \leq x \leq 1, \quad F'(x) = \alpha (1 - x^{\alpha-1}) \geq 0, \quad 1 \leq x < \infty.
\]

Thus we have that \( F(x) \) is monotonically decreasing on \([0,1]\) with \( 0 \leq F(x) \leq 1 - \alpha \) and monotonically increasing on \([1, \infty)\) with \( F(x) \geq 0 \).

Since \( F(x) \) is a nonnegative function on the interval, by the standard operational calculus we have for a positive operator \( T \) on \( H \) and \( 0 \leq \alpha \leq 1 \):

\[
\alpha T + 1 - \alpha \geq T^\alpha. \tag{2}
\]

By replacing \( T \) with \( T^{-1} \) in (2) we get

\[
\alpha T^{-1} + 1 - \alpha \geq T^{-\alpha}, \tag{3}
\]

and by taking inverses of both sides (3) we get

\[
T^\alpha \geq (\alpha T^{-1} + 1 - \alpha)^{-1}. \tag{4}
\]

Combining (2) and (4) we conclude

\[
\alpha T + 1 - \alpha \geq T^\alpha \geq (\alpha T^{-1} + 1 - \alpha)^{-1}. \tag{5}
\]
Putting $T = A^{-1/2} B A^{-1/2}$ in (5) we have
\[
\alpha A^{-1/2} B A^{-1/2} + 1 - \alpha \geq (A^{-1/2} B A^{-1/2})^a \geq [\alpha (A^{-1/2} B A^{-1/2})^{-1} + 1 - \alpha]^{-1}.
\] (6)

Finally, multiplying by $A^{1/2}$ on both sides in (6) we achieve
\[
\alpha B + A - \alpha A \geq A^{1/2} (A^{-1/2} B A^{-1/2})^a A^{1/2} \geq A^{1/2} [\alpha (A^{-1/2} B A^{-1/2})^{-1} + 1 - \alpha]^{-1} A^{1/2},
\]
\[
\alpha B + A - \alpha A \geq A^{1/2} (A^{-1/2} B A^{-1/2})^a A^{1/2} \geq (\alpha B^{-1} + A^{-1} - \alpha A^{-1})^{-1},
\]
\[
(1 - \alpha) A + \alpha B \geq A^{1/2} (A^{-1/2} B A^{-1/2})^a A^{1/2} \geq [(1 - \alpha) A^{-1} + \alpha B^{-1}]^{-1}.
\]

**Theorem 2**

Let $T$ be a positive invertible operator on a Hilbert space $H$. Then the following hold and are mutually equivalent:

(i) If $1 \geq \alpha \geq 0$ then $\alpha T + 1 - \alpha \geq T^a$;

(ii) if $\alpha > 1$ then $\alpha T + 1 - \alpha \leq T^a$;

(iii) if $\alpha < 0$ then $\alpha T + 1 - \alpha \leq T^a$.

**Proof**

Obviously assertion (i) was already obtained in (2). The rest is to show the equivalence of (i), (ii) and (iii).

(\text{i}) $\Leftrightarrow$ (\text{ii}): Suppose $\alpha > 1$. Assertion (i) is equivalent to $S^{1/\alpha} \leq (1/\alpha) S + 1 - (1/\alpha)$, i.e., $\alpha S^{1/\alpha} \leq S + \alpha - 1$. Putting $T = S^{1/\alpha}$ we have $T^\alpha \geq \alpha S + 1 - \alpha$. Thus (i) implies (ii). Similarly we can get (ii) implies (i).

(\text{ii}) $\Leftrightarrow$ (\text{iii}): Assertion (ii) is equivalent to $\alpha S + 1 - \alpha \leq S^a$. Multiply both sides of this inequality by $S^{-1/2}$ to obtain the equivalent inequality $\alpha + (1 - \alpha) S^{-1} \leq S^{a-1}$ for any $\alpha > 1$. Then set $\lambda = 1 - \alpha < 0$ and $T = S^{-1}$. We get $\lambda T + 1 - \lambda \leq T^\lambda$. Thus (ii) implies (iii). Similarly we can get (iii) implies (ii).
Theorem 3

Let $A$ and $B$ be positive invertible operators on a Hilbert space $H$. Then the following hold and are mutually equivalent:

(i) If $1 \geq \alpha \geq 0$ then $(1-\alpha)A + \alpha B \geq A^{1/2}(A^{-1/2}BA^{-1/2})^\alpha A^{1/2}$;

(ii) if $\alpha > 1$ then $(1-\alpha)A + \alpha B \leq A^{1/2}(A^{-1/2}BA^{-1/2})^\alpha A^{1/2}$

(iii) if $\alpha < 0$ then $(1-\alpha)A + \alpha B \leq A^{1/2}(A^{-1/2}BA^{-1/2})^\alpha A^{1/2}$.

Proof

Use Theorem 2 by putting $T = A^{-1/2}BA^{-1/2}$ we have

(i) If $1 \geq \alpha \geq 0$ then $\alpha A^{-1/2}BA^{-1/2} + 1-\alpha \geq (A^{-1/2}BA^{-1/2})^\alpha$;

(ii) if $\alpha > 1$ then $\alpha A^{-1/2}BA^{-1/2} + 1-\alpha \leq (A^{-1/2}BA^{-1/2})^\alpha$;

(iii) if $\alpha < 0$ then $\alpha A^{-1/2}BA^{-1/2} + 1-\alpha \leq (A^{-1/2}BA^{-1/2})^\alpha$;

and then multiplying (9) by $A^{1/2}$ on both sides we get

(i) If $1 \geq \alpha \geq 0$ then $(1-\alpha)A + \alpha B \geq A^{1/2}(A^{-1/2}BA^{-1/2})^\alpha A^{1/2}$;

(ii) if $\alpha > 1$ then $(1-\alpha)A + \alpha B \leq A^{1/2}(A^{-1/2}BA^{-1/2})^\alpha A^{1/2}$;

(iii) if $\alpha < 0$ then $(1-\alpha)A + \alpha B \leq A^{1/2}(A^{-1/2}BA^{-1/2})^\alpha A^{1/2}$;

as desired.

CONCLUSION

Inequalities in (5) are very important and fundamental. Inequalities in (1) in Theorem 1 can be obtained from (5) by putting $T = A^{-1/2}BA^{-1/2}$. Statements in Theorem 2 can be derived from the first inequality in (5). Statements in Theorem 3 obviously can be derived from statements in Theorem 2 by putting $T = A^{-1/2}BA^{-1/2}$ and then multiplying by $A^{1/2}$ on both sides.
REFERENCES

